

The Mathematics of Financial Planning
(supplementary lesson notes to accompany FMGT 2820)

Using the Texas Instruments BA II Plus Calculator

Reference is made to the Appendix Tables A-1 to A-4 in the course textbook
Investments: Analysis and Management, Second Canadian Edition, 2005.

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with the permission of:
Richard McCallum
BCIT
Phone: (604) 456-8171
e-mail: rick_mccallum@bcit.ca

Revised by Derek Knox – January, 2008

Introduction

The majority of decisions in the field of financial planning (and corporate finance for that matter) involve cash flows in different time periods. For example, one may wish to know how much they need to save each year in order to have amassed \$200,000 by the time they reach the age of 60. Or, in the event of a forced early retirement package, would it be preferable to accept a buy out consisting of a single lump sum of \$75,000 today or a series of payments of \$20,000 each year for the next 4 years? Even the values of investment instruments such as T-bills, bonds and stocks are determined using the math of finance (also known as the time value of money).

Much of financial planning involves choosing between alternative courses of action involving differing sequences of cash flows. In order to make these decisions, the receipts and payments of money in different time periods must be put on a comparable basis. And it is the mathematics of finance that allows us to do this in order that apples can be compared to apples.

In the notes that follow, the student will first be presented with a theoretical foundation that views all cash flows as being one of three types: a single lump sum today, a single lump sum at some point in the future or a series of equal payments (or receipts) that continue for a specific number of periods. Each of these types of cash flow has a special name as we shall discover. There will be two other variables involved, an interest rate and a number of time periods (normally years); math of finance problems usually involve solving for one of the variables which is unknown. While discussing the theory, solutions will be obtained in three ways:

- (a) using basic arithmetic and equations
- (b) using compound interest tables
- (c) using a financial calculator.

Once we move on to the more practical world of financial planning applications we shall concentrate on using the calculator only.

Calculators

A number of financial calculators exist which perform math of finance calculations. These notes will assume the use of a Texas Instruments BA II Plus calculator but, since all financial calculators are fairly similar, the key strokes can easily be adapted to the other calculators.

There are five main keys which are used in financial planning applications and which relate to the variables mentioned above. There are other keys but these relate to more complex calculations and will be introduced later. The main keys are located in the third row of keys from the top and, beginning from the left, are:

- n Refers to the number of periods in the problem. The periods most commonly used are years but this is not always the case. Periods can also be defined as months, half-months, days, etc. It is important to be aware of this.
- i Refers to the interest rate that is to be used in the calculation and it must be the interest rate for the period chosen. If the period is years, interest must be expressed as an annual rate; if the period is expressed in months, then the interest rate must be a monthly rate.
- PV or Present Value refers to a single sum of money as of today.
- PMT or Payment refers to one of a series of equal sums to be received (or paid) every period for n periods.
- FV or Future Value refers to a single sum of money at some point in the future (n periods into the future to be exact).

In a financial planning type of problem three (or four) of these variables will be known and the planner will be solving for the fourth (or fifth). It sounds simple but the key is being able to determine whether a particular cash flow represents a present value, future value or a payment. This ability comes with practice.

Another important calculator key is the CPT or compute key which instructs the calculator to solve for one of the five variables. For example, by pressing the CPT key followed by the PV key, the calculator will solve for the present value if the other data has been inputted correctly.

A very important operation is the clearing of the calculator between calculations. The variables which have been entered or calculated remain in memory registers in the calculator and may carry over into the next calculation giving erroneous results. The memory can be cleared by pressing the 2ND key followed by the CLR TVM key. Note that the 2ND key allows most of the other keys to perform a second function which is written in yellow lettering above each key. Thus by pressing 2ND, the FV key is converted to CLR TVM, which stands for *clear TVM* (Time Value of Money) memory registers. Note that the CE/C key only clears the screen.

Other Calculators:

Other business calculators have the same 5 variable keys but the operations to clear the TVM memory registers are different.

Sharp	
EL-733A	2ndF C.CE
EL-738	2ndF CA
Hewlett-Packard	
HP-10B	Clear All
HP-12C	Clear FIN

Theory

I Compounding (or Future Valuing)

Lump Sum

The process of compounding or calculating a future value is the operation with which most of us are familiar and so we shall begin our study of the theory of the math of financial planning here.

Let us assume that we go into our neighbourhood credit union and invest \$1,000 in a term deposit which will *mature* or come due in one year. The interest rate (sometimes referred to as the compounding rate) that will be paid on the term deposit is 4%. How much will we receive back at the end of the year? We will receive the original principal of \$1,000 plus \$40 in interest for a total of \$1,040? The \$40 is calculated as 4% (or .04) times \$1,000. If we had taken out a 2 year term deposit instead, also at 4%, then at the end of 2 years we would receive \$1,081.60 — calculated as \$1,000 plus \$40 interest in year one plus \$41.60 interest in year two. The year two interest is calculated as 4% times \$1,040 — that is, it is based on the initial principal plus the interest that had been earned during the first year. This illustrates the concept of *compound interest*, where interest is earned on interest, as opposed to *simple interest* in which the interest during the second year would be the same as the first year (\$40) since it is based on the initial principal. Simple interest is rarely used in financial planning and so any reference to interest in the remainder of these notes will assume compound interest.

Another way of describing the above transaction is to say that the future value of \$1,000 (at 4%) in one year is \$1,040 and in two years is \$1081.60. These values, and the future values for any time period and any interest rate, can be arrived at mathematically, using tables, or using the financial functions on a business calculator. These techniques will now be illustrated.

(a) *Mathematically*

We could have made the above computations mathematically. Note that * represents the multiplication operation.

One Year: $\$1,000 * 1.04 = \$1,040$ where 1.04 is 1 plus the interest rate expressed as a decimal.

Two Years: $\$1,000 * 1.04 * 1.04 = \$1,081.60$ or $\$1,000 * (1.04)^2$.

Ten Years: $\$1,000 * (1.04)^{10} = \$1,480.24$.

Some calculators have keys that allow one to raise a number to a power (1.04 raised to the 10th power) but the financial functions ((c) below) are a more efficient way of using the calculator.

(b) *Tables*

There are four tables of factors that can be used in financial planning calculations to replace the above mathematics. These are found as appendices in the text following page 723. Table A-1, titled Compound (Future) Value Factors for \$1, will accomplish the same as the above but without needing to raise a number to a power. In the calculations which follow F1 will be used to represent a factor from Table A1, F2 will represent a factor from Table A2, etc.

To find the future value of \$1,000 in two years at 4%, it is necessary to scan across Table A-1 until we see 4% and then go down the table until we see the number of years: 2. The resulting factor is 1.082. The calculation is $\$1,000 * 1.082 = \$1,082.00$. Note that these tables only carry the numbers to three decimal places of accuracy.

As another example, calculate the future value of \$500 in seven years, at 6%. The factor is 1.504 times \$500 equals \$752.00.

(c) *Financial Calculator*

The most efficient way by far to perform these calculations is using a financial calculator. The key strokes to solve the original question, future value of \$1,000 in two years at 4%, are:

2ND CLR TVM (to ensure that memory is cleared).
1000 PV
2 n
4 i
CPT FV

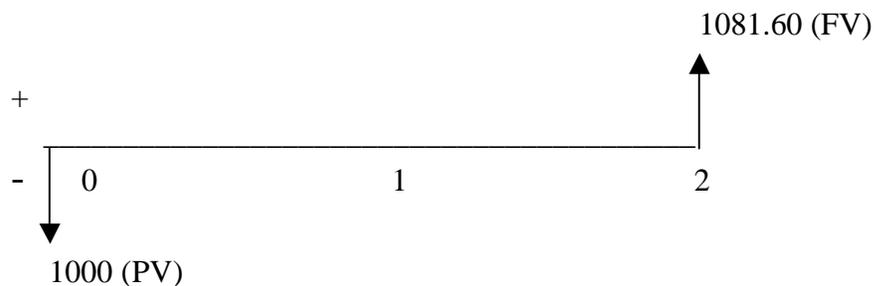
and the answer that appears on the screen is negative \$1081.6. To find out what the \$1,000 would grow to in 5 years, it is necessary only to key in a new value for n (by pressing 5 n) and then COMP FV. The result will be negative \$1216.65 (rounded). This answer (of \$1,216.65) is the maturity amount of a five-year term deposit with an interest rate of 4%. Notice that it was not necessary to clear the memory and re-enter the data for the second calculation since only one variable was changed. However this shortcut should be used with caution. You must be certain that you want the memory registers to have the values that remain in them. Sometimes it is safer to clear the memory and key in all the variables again.

Several other points should be noted. First, the order in which the data is entered is not important. We can solve for any of the four variables if the other three are known. For instance, if we wanted to know what interest rate would cause a deposit of \$100 to grow to \$200 in 10 years we would be solving for i. This will be illustrated on page 16. The second point to note is that the FV that results is a negative number. The Texas Instruments calculator assumes that this will be opposite in sign to the present value. This will be discussed in more detail when we introduce cash flow diagrams in the next section.

This process of finding the future value of a single amount (or lump sum) is the simplest type of math of financial planning calculation. Now we shall consider the problem of finding the future value of a stream of equal cash flows.

Annuity

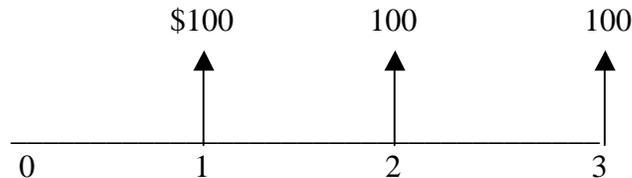
An annuity is defined as a stream of equal cash flows for a certain number of periods. These cash flows can be positive (receipts from a pension) or negative (loan payments). Before continuing with annuities it is useful to adopt a convention for visually portraying cash flows. Often the biggest problem with financial planning calculations is determining the cash flows and when they take place. Cash flows will be shown on a time line with cash outflows below the line and cash inflows above the line. The line will begin at time zero (today) and dollar amounts that appear at time zero are called present values. The earlier example, investing in a two year term deposit, would be represented as:



In this case the initial investment is shown as negative since it represents a cash outflow from the investor's viewpoint and the FV (\$1,081.60) is shown as a positive return to the

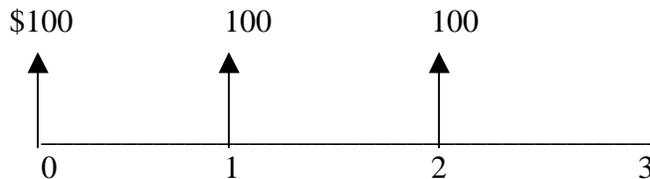
investor. So to be absolutely correct we should have entered the \$1000 as a negative present value and the result would have been a positive future value. In practice, with these types of simple calculations, the present value is usually entered as a positive number and we just ignore the negative sign in the resulting future value. It saves a keystroke. *This convention will be assumed in the remainder of these notes.*

These time lines are the best way to illustrate the difference between the two types of annuity, a *regular annuity* and an *annuity due*. A three year *regular annuity* of \$100 would be represented as:



As an example consider the problem of how much will I have saved after three years if I contribute \$100 each year into a savings account which pays 10% interest and if my first contribution takes place one year from now.

An *annuity due* would still have three equal contributions of \$100 but the first one would take place today. The time line representation is:



As before, the future values can be solved three ways:

(a) *Mathematically*

The mathematical solution of the regular annuity case can be done using three separate future value calculations. The first \$100 contribution is made at time 1 and so it only compounds for two periods. The second contribution compounds for one period and the third contribution is made at time 3 so it does not collect any interest at all.

$$\begin{array}{rcl}
 \text{Future value} = & \$100 * (1.10)^2 = & \$121.00 \\
 \text{plus} & 100 * (1.10) = & 110.00 \\
 \text{plus} & 100 & = \underline{100.00} \\
 & & \$ 331.00
 \end{array}$$

and for an annuity due the calculation is:

$$\begin{array}{r} \text{plus} \quad \$100 * (1.10)^3 = \$ 133.10 \\ \text{plus} \quad 100 * (1.10)^2 = 121.00 \\ \text{plus} \quad 100 * (1.10) = \underline{110.00} \\ \quad \quad \quad \$ 364.10 \end{array}$$

Note that even though both annuities consist of three payments, the future value of an annuity due *is greater* than that of a regular annuity because the contributions into the annuity due are made earlier and therefore compound for a longer period of time.

Another point to notice is that the mathematical technique for calculating annuity future values becomes increasingly tedious as the number of periods increases. Consequently, we shall move to the other two approaches.

(b) *Tables*

In order to find the future value of a regular annuity, it is necessary to use Table A-3, titled Future Value Annuity Factors for \$1. As previously, we go across the table to the desired interest rate (10%) and then down the table for the number of periods (3) and we find the factor 3.31. This is multiplied by the amount of the annuity (\$100) to come up with a future value of \$331.00. The future value of a ten year annuity of \$500 at 8% would be $\$500 * 14.487 = \7243.50 .

To solve for an annuity due it is necessary to multiply the result for a regular annuity by 1 plus the interest rate. Considering the original three-year annuity due of \$100, we take the prior result of \$331 and multiply by 1.10 and the result is \$364.10 as calculated mathematically. We won't spend further time with the tables because again the calculator is much easier.

(c) *Calculator*

The key strokes to solve for the three year regular annuity of \$100 are

```
2ND CLR TVM
100 PMT
3 n
10 i
CPT FV
```

and \$331.00 will appear on the screen. If the annuity is an annuity due it is first necessary to activate the BGN function (2ND BGN followed by 2ND SET). This will result in a small BGN appearing at the top of the screen. When this BGN is showing all annuity calculations done will assume an annuity due; that is, the first

cash flow takes place at time 0. To return to the regular annuity mode, just press 2ND BGN 2ND SET and the BGN at the top of the screen will disappear.

II) Discounting (or Present Valuing)

The second major type of mathematical calculation in financial planning is that of discounting or finding the present value of a dollar amount that will be received in the future. The process of discounting seems to be more difficult to conceptualize than compounding for several reasons. First, the types of financial instruments with which the average investor is most familiar (savings accounts, term deposits) involve compounding. However, as you proceed in the field of financial planning, you will learn that the valuation of most financial instruments (such as stocks and bonds) and most decision making require the use of present values. Second, whereas there is usually an obvious interest rate associated with a compounding situation such as the interest rate on a savings account, this is not true with a discounting situation. Rather we usually have to rely on an interest rate that is referred to as an *opportunity cost*. That is, it is the interest rate (or rate of return) that would be available to the investor or decision maker from the most promising alternative that is *not selected*. By deciding on a particular investment or course of action, all others are rejected and thus so is the rate of return that could have been earned on the best of the rejects. Fortunately, in most textbook examples this interest rate will be given but in practice it may be more difficult to determine.

Lump Sum

Let us say that you have just won \$10,000 in a lottery but the terms of the lottery state that you will not receive your pay out until one year from now. A friend has offered to buy the ticket from you and pay you today. However you must agree on a price and this is done by finding the present value of the \$10,000. In order to do this you must determine the appropriate interest (or discount) rate to use. We'll use 6% but in practice you might look at the going rate of interest on a one year term deposit and use that. Similar to compounding, we can solve for present values mathematically, using tables, or using a financial calculator.

(a) *Mathematically*

We can use the relationship that was developed to calculate future values
Future value = present value * 1.06 and rearrange it so that

$$\text{Present value} = \frac{\text{Future value}}{1.06} = \frac{10,000}{1.06} = \$9,433.96$$

So we would be willing to sell our lottery ticket for \$9,433.96 which is the present value of \$10,000 taken back one year at 6%.

If the \$10,000 was to be received two years from now the present value would be

$$\frac{10,000}{(1.06)^2} = \$8,899.96$$

To confirm your work, calculate to what sum \$8,899.96 would grow in two years at 6%

$$\text{Future value} = \$8,899.96 * (1.06)^2 = \$10,000.$$

(b) *Tables*

Table A-2, titled Present Value Factors for \$1, will accomplish the same as the above calculation. Scan across this table to 6% and then go down to the appropriate number of years — 1. The factor is .943 which is then multiplied by the future value of \$10,000 to arrive at \$9,430. The discrepancy from the mathematical answer arises because factors are only displayed to three decimal places of accuracy. Notice that all factors in this table are smaller than 1 which means that a *present value will always be smaller* than its associated future value. To find the present value using a different discount rate or number of years it is only necessary to search the table for the appropriate factor.

(c) *Calculator*

Again the most efficient way to compute a present value is with a financial calculator.

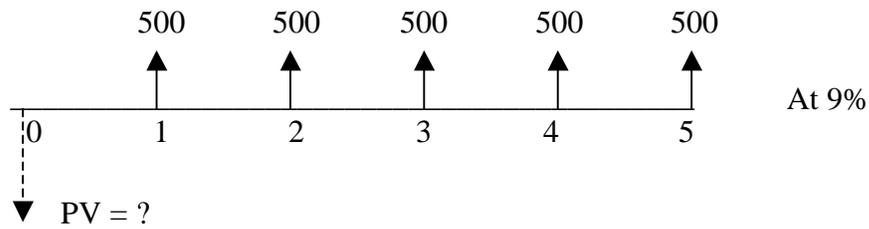
The key strokes are:

```
2ND CLR TVM
10000 FV
1 n
6 i
CPT PV
```

and the answer that appears on the screen is negative 9,433.96. To find the present value for 2 years, key in 2 as the new value of n followed by CPT PV and the result of 8,899.96 will be displayed.

Annuity

The present value of the annuity shown on the following page can also be solved using either of the three methods (math, tables, calculator) but we shall consider only the last two.



(b) *Tables*

The table to be used is Table A-4, Present Value Annuity Factors for \$1 per period. Going across to 9% and down to 5 years, we find the factor 3.89 which, multiplied by \$500, results in a present value of \$1,945.

(c) *Calculator*

The key strokes to solve this problem are

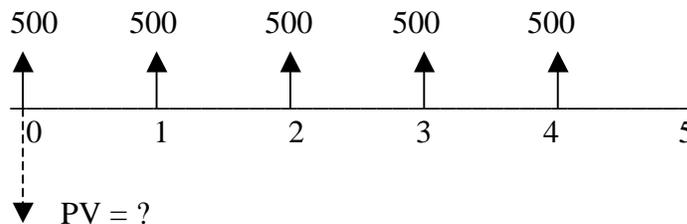
```

2ND CLR TVM
500 PMT
5 n
9 i
CPT PV

```

and negative \$1,944.83 will be displayed on the calculator. Once again the difference is due to the limited accuracy of the tables.

These examples have been regular annuities since the first cash flow begins one period hence. If the first cash flow takes place immediately (at time = 0) as diagrammed below, it will be necessary to activate the annuity due function by depressing the 2ND BGN 2ND SET keys.



The present value of this 5 year annuity due at 9% is \$2,119.86. Two points are worth noting: first, the present value of an annuity due *is larger* than the present value of a regular annuity (\$2,119.86 vs \$1,944.83) since the first cash flow is already a present value and therefore does not need to be discounted; second, the last cash flow associated with the annuity due takes place at the beginning of year 5 which can also be considered to be the end of year 4.

Perpetuity

A perpetuity is a special type of annuity. It consists of a constant cash flow received (or paid) at regular intervals but it continues forever. Conceptually the present value of a perpetuity could be calculated by using a large value for n. For example, the present value of a \$100 perpetuity at an interest rate of 8% could be obtained by

```
2ND CLR TVM
100 PMT
999 n
8 i
CPT PV
```

and the result would be \$1,250. However this same result can be obtained by dividing the amount of the perpetuity by the discount rate expressed as a decimal. That is,

$$\frac{\$100}{.08} = \$1,250$$

Growing Perpetuity

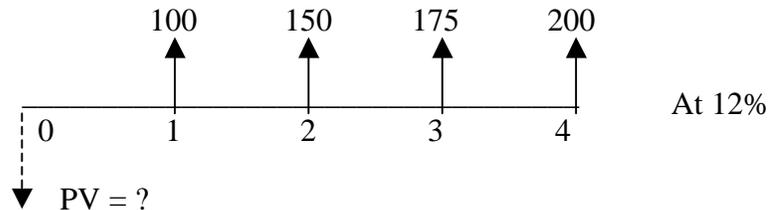
One other special case needs to be considered; that of a cash flow that goes on forever but instead of remaining a constant amount, it *grows* at a *constant rate* forever. Using the above example of \$100 cash flow to be received one year from now and assuming that it will grow at a rate of 2% forever, the formula for finding the present value of this series is

$$\text{Present value} = \frac{\text{Cash Flow (beginning in 1 year)}}{(\text{discount rate}) \text{ minus } (\text{growth rate})} = \frac{\$100}{.08 - .02} = \$1,666.67$$

Notice that both the discount rate and the growth rate must be expressed as decimals to be used in the calculation. The importance of this formula will be evident as we move into the next section and use math of finance techniques to place a value on the various financial assets.

Irregular Cash Flows

Sometimes it will be necessary to find the present value of a more complicated series of cash flows such as shown below



This could be solved with the calculator as four separate calculations in which the lump sums are present valued for one, two, three and four years respectively and then added together. However the BA II plus calculator has keys which will perform this type of calculation very easily. These are the CF and the DATA (ENT) key found on the rightmost column of keys. CF stands for Cash Flow period 1, period 2, etc. It is similar in use to the PMT key in that it allows the entry of multiple cash flows but differs in that the cash flows do not need to be equal. Solving for a present value using the CF keys requires three steps – first the calculator must be cleared, then the data must be entered and finally the calculation made.

Step 1 – Clearing the calculator (note this is a different *clear* command than we saw earlier)

Press the CF key

Then press 2ND CLR WORK

At this point the screen will display $CF_0 = 0$

Step 2 – Enter the data

0 ENTER then down arrow and CO_1 appears in the screen

100 ENTER then down arrow and FO_1 appears in the screen

1 ENTER this enters the frequency of this cash flow (once)

Continue with the down arrows and enter the cash flows for periods 2 to 4

150 ENTER

1 ENTER

175 ENTER

1 ENTER

200 ENTER

1 ENTER

Step 3 – Calculating

Press the NPV key and $I = 0$ will appear on the screen

12 ENT (to enter the interest or discount rate)

Press Down Arrow and $NPV = 0$ will appear on the screen

Press CPT and the answer 460.53 will appear

Notice that the first cash flow prompt (CF_0) requires the entry of zero since there is no cash flow at time zero. This will change in other situations where there is an initial investment which must be entered as a negative number for CF_0 .

Practice

Now that the basic theory of the mathematics of financial planning has been covered we shall turn to typical problems that the financial planner faces. In this section we will be relying only on the financial calculator. The keystrokes to clear the calculator will not be shown but it is important that this operation be performed at the beginning of each calculation.

Future Sums

The simplest situation can be articulated as “if I save \$2,000 per year and my savings earn interest at the rate of 4%, how much will I have after 25 years?” The key strokes to solve this are:

```
2000 PMT
4      i
25     n
CPT FV → 83,291.82
```

More likely you will have a specific goal in mind and want to know what amount must be saved annually to accumulate this amount. For example, how much must I save each year for the next 5 years if I want to accumulate \$50,000 and my savings earn interest at the rate of 8%.

```
50000 FV
5      n
8      i
CPT PMT → 8,522.82
```

The assumption here is that the first deposit (of 8,522.82) will take place at the end of the first year and the \$50,000 will be available at the end of the fifth year. If the first deposit is to take place today (at time 0) it is necessary to turn on the BGN function and recalculate (or key in the data again if the calculator has been cleared). The annual savings in this case would be 7,891.50. The important lesson is that the earlier you start saving the better. At this point we are only considering annual cash flows but this approach can easily be adapted to monthly or weekly savings as we shall do on page 22.

Practice problems:

1. You are planning to take a sabbatical from your work in 5 years time and want to accumulate \$30,000 by then in order to travel. How much must you set aside each year (starting one year from now) to reach your goal? Assume an interest rate of 5%.

Answer: \$5,429.24

2. You wish to make a *single* deposit today that will earn interest at the rate of 6% and that will grow to \$14,000 in 8 years time when your daughter enters college. How much must this deposit be?

Answer: \$8,783.77

3. Al Arbour has just purchased a new house for \$175,000. If house prices continue to rise at a rate of 3% per annum, how much will Al's house be worth in 10 years time?

Answer: \$ 235,185.37

Loan Amortization

As an example of this type of calculation, let's assume that we have just taken out a \$25,000 loan. The loan has an interest rate of 14% and will be repaid by means of 4 equal annual payments beginning one year from now. What is the amount of these annual payments?

25000 PV
4 n
14 i
Comp PMT → 8,580.12

It may also be necessary to break this payment (of \$8,580.12) into an interest and principal component each year in order to complete a tax return since interest may be a tax deductible expense (if the loan is for investment purposes) while the principle repayment is never tax deductible. The table below shows this breakdown.

<u>Year</u>	<u>Opening Balance</u>	<u>Payment</u>	<u>Interest</u>	<u>Principal Repayment</u>
1	25,000	8,580.12	3,500.00	5,080.12
2	19,919.88	8,580.12	2,788.78	5,791.34
3	14,128.54	8,580.12	1,978.00	6,602.12
4	7,526.42	8,580.12	1,053.70	7,526.42

The interest column is calculated as opening balance * 14% (25,000 * .14 = 3,500.00).

The principal repayment is calculated as payment less interest ($8,580.12 - 3,500 = 5,080.12$).

The opening balance in year two is calculated as the opening balance in year one less the principal repayment in year 1 ($25,000 - 5,080.12 = 19,919.88$).

As you might imagine, this calculation can get a bit tedious but fortunately the Sharp calculator has keys that will perform the amortization for us. Once the initial calculation (COMP PMT → 8,580.12) has been completed, **and without clearing the calculator**, simply press the AMRT key and the prompt P1 will appear. Enter the period number (say 3) followed by the ENTER key. Press the down arrow key and the prompt P2 will appear. Enter 3 (the same number) followed by ENTER. Then press the down arrow three times: the first depression gives you the balance at the end of period 3 (\$7,526.42); the second depression gives you the principal repayment in period 3 (\$6,602.12) and the third depression gives you the interest payment for period 3 (\$1,978.00). You can then go on to enter another period and repeat the calculation. Also note that you can perform the same calculation over multiple periods by entering the appropriate P1 and P2.

Most personal loan arrangements require monthly payments rather than annual. This complicates matters slightly and will be covered in a later section, More Frequent Compounding (page 23).

Practice Problems:

1. What are the annual payments necessary to pay off a \$7,500 loan over 5 years at an interest rate of 8%? Also, break the year 4 payment into its principal reduction and interest components.

Answer: \$1878.42, 1610.45, 267.98

2. You have accumulated \$150,000 in an RRSP and now wish to use the funds to purchase a 20 year annuity with the first payment beginning one year from now. What would be the annual receipt from the annuity if the going interest rate is 7%?

Answer: \$14,158.94

Finding an Unknown Interest Rate

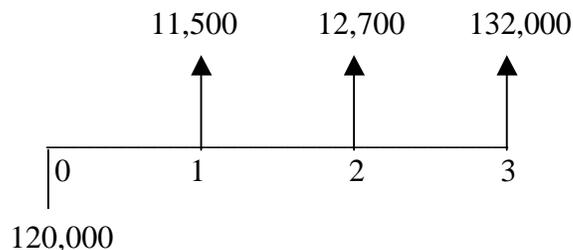
Another type of calculation involves the determination of an unknown interest rate. For example, if a person is able to save \$2,500 per year starting one year from now, what interest rate must be earned in order to accumulate \$40,000 in 10 years.

-2500 PMT (the negative is entered using the +/- key)
40000 FV

10 n
CPT i → 10.08%

Notice that in this case the PMT and FV were of opposite signs. It does not matter which is positive and which is negative, but they cannot both have the same sign if you are solving for the interest rate (or for the number of periods). Another thing to notice is that your calculator paused 5 to 10 seconds with a blank screen while performing this calculation. This occurs because the calculator is actually using a trial and error process to arrive at an answer.

If the relevant cash flows are not regular (that is, either a single lump sum or an annuity) then it will be necessary to use the CF and IRR function. For example, say we had invested \$120,000 in a rental property from which we received revenues of \$11,500 and \$12,700 at the end of the first two years and then we sell the property for \$132,000 at the end of the third year. The cash flow could be represented as shown below:



The key strokes to calculate the rate of return that we received from this investment will be similar to those used on page 13 except that there is now a cash flow at time zero and we will be calculating IRR (which stands for internal rate of return).

Step 1 – Clearing the calculator

Press the CF key
Then press 2ND CLR WORK
At this point the screen will display $CF_0 = 0$

Step 2 – Enter the data

120000 +/- ENTER then down arrow and CO_1 appears in the screen
11500 ENTER then down arrow and FO_1 appears in the screen
1 ENTER
Continue with the down arrows and enter the cash flows for periods 2 to 3
12700 ENTER
1 ENTER
132000 ENTER
1 ENTER

Step 3 – Calculating

Press the IRR key and I = 0 will appear on the screen

Press CPT and the answer 10.04 will appear

Practice Problems

1. William Bennett bought a piece of land for \$3,000 six years ago. He just sold it for \$5,200. What rate of return did he make on his investment?

Answer: 9.6%

2. A share of stock has doubled in value during the last five years. At what (annual) rate has it been growing?

Answer: 14.87%

3. How long would it take you to accumulate \$5,000 if you saved \$800 per year (starting one year from now) at an interest rate of 5%? How long if the interest rate was 9%?

Answer: 5.57 years; 5.18 years.

4. Alf Leigh bought 100 shares of stock five years ago for a total of \$750 and has just sold the stock for \$1125. During the time that he held the stock Alf received dividend payments of \$65 per year for the first three years and \$90 per year for the last two. The second \$90 was received on the same day that Alf sold the stock — that is, at the end of the fifth year. What rate of return did Alf earn on his investment?

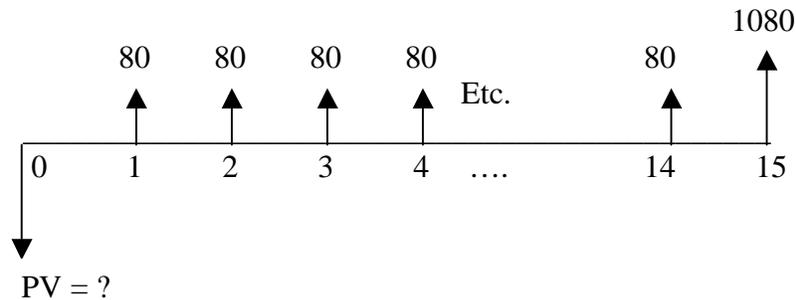
Answer: 16.85%

Bond Values

A bond is a type of investment which pays a regular periodic interest payment and then returns to the investor the *par value* or *maturity value* on a date some considerable distance in the future. Although the periodic interest payments are usually received semi-annually, we shall assume that they are received annually to simplify the concept. These payments are specified by the *coupon rate* associated with the bond.

Let's assume a thousand dollar par value bond with a coupon rate of 8% that *matures* or comes due in 15 years time. The annual payments would be \$80 (8% * \$1000) and these would be received each year for the next fifteen years. In addition the bondholder would

also receive the par value (\$1,000) at the end of the fifteenth year. Therefore this bond would provide the cash flows shown below to the holder of the bond:



The question of how much to pay in order to purchase this bond then comes down to a present value calculation in which these cash flows are present valued at the investor's *required rate of return*. This required rate of return is based on the general level of interest rates in the economy and on the perception of *risk* associated with this bond. For the purpose of discussing the mechanics of financial planning mathematics this rate will be given (assume 11%) but the concept of a required rate of return for a particular investment is important and will be studied elsewhere in the course.

The present value calculation could be performed using the CF key but a simpler method is as follows:

```

1000  FV
  80  PMT
  15  n
  11  i
CPT PV → 784.27

```

\$784.27 then is the amount that an investor would pay today for this bond if the investor required a rate of return of 11%. If the investor required a rate of return of 12%, he/she would only be willing to pay \$727.57. You can prove this by redoing the above calculation and substituting 12 for the *i*. Another point to note with the key entries above is that with most financial calculators it is possible to enter both the FV and PMT in order to calculate the PV.

A related calculation involving bonds that the financial planner might face is the case where the current market price is given for a particular bond and the rate of return must be calculated. If we assume the same bond as described above but we are told that it has a market price of \$750, then what rate of return must *the market* (the average investor) be requiring? The keystrokes are:

```

750 +/-    PV

```

1000	FV
80	PMT
15	n
COMP i	→ 11.59%

This required rate of return (of 11.59%) is also known as the *yield to maturity* on the bond since it is the yield or return that an investor would receive from purchasing the bond for \$750 and holding it until it matures in 15 years time.

Note that the sign of the dollar amounts entered for the FV and the PMT must be the same and opposite to the sign of the dollar amount of the PV.

Practice Problems

1. At what price would a \$1,000 par value, 4% coupon bond with 7 years until maturity sell for if the required rate of return is 7%?

Answer: \$838.32

2. How much would you be willing to pay for a \$1,000 par value, 12% coupon bond with 18 years to maturity if your required rate of return is 9.5%?

Answer: \$1,211.78

3. What is the yield to maturity on a \$1,000 par value, 8% coupon bond with 6 years remaining until maturity if it is currently selling for \$895?

Answer: 10.44%.

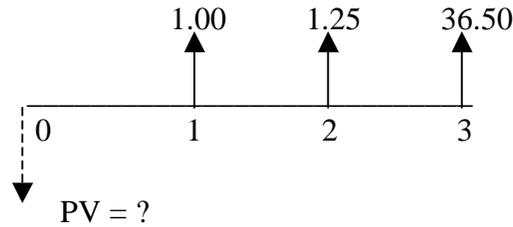
Common Stock Values

Common stock represent the *ownership interest* in a corporation and, as such, the return to common shareholders is not guaranteed but rather consists of a share of the corporate profits (or losses). This share of profits paid out to common shareholders is known as a *dividend*. The problem of placing a value on a share of common stock and the related problem of predicting changes in value or price for shares are among the most difficult in financial planning. We shall look at several simple approaches in order to provide more practice in the math of financial planning.

Case 1

We want to know how much a stock is worth (or we would be willing to pay for one share) if we plan to hold the stock for a defined length of time (say 3 years) after which

time we assume that we can sell the stock for \$35.00. Also, during the time we are holding the stock we expect to receive dividends of \$1.00, \$1.25, and \$1.50 at the end of each of the three years. Our required rate of return is 18%. Valuation of the stock is simply a matter of present valuing the following cash flows:



Step 1 – Clearing the calculator

Press the CF key
 Then press 2ND CLR WORK
 At this point the screen will display $CF_0 = 0$

Step 2 – Enter the data

0 ENTER then down arrow
 1.00 ENTER then down arrow
 1 ENTER etc.
 1.25 ENTER
 1 ENTER
 36.50 ENTER
 1 ENTER

Step 3 – Calculating

Press the NPV key and $I = 0$ will appear on the screen
 18 ENT (to enter the interest or discount rate)
 Press Down Arrow and $NPV = 0$ will appear on the screen
 Press CPT and the answer 23.96 will appear

From this we can say that this stock has a value today of \$23.96. Several points should be noted:

First, the investor's required rate of return is not easy to estimate but, as before, is a function of the perceived risk of the cash flows which in turn is determined by this company's prospects for success or profitability. This will always be *less certain* than the cash flows that would result from the same corporation's bonds and so the required rate of return is almost always higher for common stock than for bonds or any of the other investment instruments that we have seen.

Second, how did we arrive at the estimated stock price of \$35.00 in three years time? This is a difficult figure to objectively determine so many analysts pose the problem in a slightly different (but not unrelated) way as we will see in Case 2.

Case 2

With this approach (favoured more by academics) the analyst focuses on the expected growth rate of a corporation and then treats the cash flows that result from owning one share of stock as a *growing perpetuity*. This allows the use of the formula given on page 12 and also does not require us to assume some arbitrary holding period. Let's say that the stock has just paid out a dividend of \$1.00 and that we expect the company to grow at an annualized rate of 9%. The present value of this cash flow, using the same required rate of return of 18% as Case 1 would be

$$\frac{(1.00) * (1.09)}{(.18 - .09)} = \$12.11.$$

Note that \$1.00 was multiplied by 1.09 in the numerator in order to convert it to the estimated dividend in one year's time.

Although forecasting growth is not a simple task, there are established techniques for doing this and so this approach is more commonly used when trying to estimate stock values. It also allows us to highlight the *two main determinants* of stock values — expected *growth* and *risk*. Anything that increases growth rates or reduces risk (and therefore required rate of return) will result in a higher stock price. Try this in the example above by redoing the calculation after increasing the growth rate by 1% or decreasing the required rate of return by 1%.

It is necessary to reemphasize the fact that valuing common stock is very difficult and the above are only two examples of many different approaches that are used in practice. The intent was to provide more examples of the math of financial planning calculations and not to make the student an expert security analyst.

Practice Problems:

1. If you demand a 14% return on your investment, how much would you be willing to pay for a share of stock that you expect to pay dividends of \$3.00, \$3.50, \$4.00, \$5.00, and \$5.75 over the next 5 years and that you expect to be able to sell for \$95.00 at the end of the fifth year.

Answer: \$63.31

2. If you paid \$22.50 for a share of stock, held it for three years during which it paid dividends of \$1.50, \$1.50, \$1.75 and then sold it for \$25.00, what rate of return did you receive on your investment?

Answer: 10.35%

3. You would like to place a value on a share of common stock for which you have the following information or estimates: growth rate of 4%, required rate of return 15%, and the company just paid a dividend of \$0.75 per share.

Answer: \$7.09

4. Use the information in Question #3 above, but now assume that a new product has been developed which will result in a higher growth rate. The new growth rate will be 7%. What will be the new share price?

Answer: \$10.03

5. Again use the basic information in Question #3 above, but now assume that as a result of NAFTA and expected increased competition from the United States, investors view the industry as being more risky and consequently require a higher rate of return — 16%. What will happen to the share price?

Answer: \$6.50

More Frequent Compounding

In the examples that we have seen so far compounding (or discounting) was assumed to take place annually. While annual compounding is most commonly assumed when doing financial planning calculations there are many situations in which the compounding period is less than one year. To illustrate, let us take a \$100 deposit at an interest rate of 12% and assume that compounding takes place annually. After one year this \$100 will

have grown to \$112. Using math, the calculation is $\$100 * (1 + .12)$ or using a financial calculator the keystrokes are:

100 PV
12 i
1 n
CPT FV → 112

Using the same example, we'll now assume that the compounding takes place **semiannually**. The 12% is the *stated* or *nominal* interest rate (also called the *annual percentage rate* or *APR*) and this must be divided by the number of periods in a year (two semiannual periods) in order to determine the *period* interest rate which will be 6%. The calculation now becomes $\$100 * (1 + .06) * (1 + .06) = \112.36 or, on a calculator:

100 PV
6 i
2 n
CPT FV → 112.36

If compounding takes place **monthly** the calculator keystrokes are:

100 PV
1 i (12% divided by 12 periods)
12 n
CPT FV → 112.68

And if we wanted to determine the future value at the end of three years with monthly compounding, the keystrokes would be:

100 PV
1 i
36 n (3 years multiplied by 12 periods per year)
CPT FV → 143.08

Notice that the more often that a sum is assumed compounded (or the *shorter the compound period*), the greater will be the future value. The opposite would be true with a discounting (present valuing) problem — the more often that a sum is assumed compounded, the smaller the present value.

The most common financial planning applications involving compounding periods other than annual are those that require the calculation of *effective annual interest rates* and *mortgage payments*. These will be examined now.

Mortgage Calculations

All fixed-rate mortgages in Canada assume semiannual compounding but the payments will be made more often than twice a year — usually these payments are made monthly, bimonthly, biweekly or weekly. Therefore, in order to calculate the amount of these payments, it is first necessary to express the semiannual rate as an effective **period** rate — whatever the period may be.

As an example we shall work out the monthly payments on a 25 year, \$125,000 mortgage with an interest rate of 8%, compounded semiannually. The keystrokes are:

125000	PV
$8 \div 2 = 4$	i
2	n
CPT FV →	135200 (intermediate answer 1)
1	n
CPT i →	8.16 (intermediate answer 2)
12	n
CPT i →	.655819692 (intermediate answer 3)
0	FV
$25 * 12 =$	n
CPT PMT →	954.02

The first series of keystrokes, as far as the intermediate answer 2 of 8.16%, is the same as seen in the previous section on EAR. The next step is to compute the effective period (in this case monthly) rate of .655819692%. Now that we have the effective interest rate for the same period for which payments are being made, the problem boils down to determining the monthly payment necessary to pay off a \$125,000 loan over 300 months (25 years times 12 months per year). Note that since \$125,000 is still present as a PV it was not necessary to reenter that amount but that FV had to be zeroed out since it contains intermediate answer 1 which is not used in the final computation. Also note that it was not necessary to compute the effective annual rate (intermediate answer 2), but rather, after answer 1, we could have gone directly to the computation of intermediate answer 3. The extra keystrokes were included in order to show the parallel with the calculations in the previous section.

Practice Problems:

1) You wish to take out a **loan** and must choose between 10.25% compounded monthly and 10.5% compounded quarterly. Compute the EAR for each and determine which is preferable.

Answer: 10.7455% (preferred) and 10.9207%

2) What would be the weekly payment on a 25 year, \$75,000 mortgage at an interest rate of 6.5% (compounded semiannually).

Answer: \$115.69